Total Dominating Sets Of Divisor Cayley Graphs

G. Mirona, Prof. B. Maheswari

Abstract— Let n $\geq$ 1 be an integer and S be the set of divisors of n .Then the set S*= {s, n- s / s $\in$ S, n $\neq$ s} is a symmetric subset of the group (Zn, $\oplus$), the additive abelian group of integers modulo n. The Cayley graph of (Zn, $\oplus$), associated with the above symmetric subset S* is called the Divisor Cayley graph and it is denoted by G(Zn , D) .That is, G(Zn , D) is the graph whose vertex set is V={0,1,2,…,n-1} and the edge set is E = {(x , y) / x-y or y-x is in S*}. Let G be a graph without isolated vertices. Then a total dominating set T is a subset of V(G) such that every vertex in V is adjacent to some vertex in T. A total dominating set with minimum cardinality is called a minimum total dominating set and the cardinality of a minimum total dominating set is called the total domination number of G and is denoted by Yt (G).

Index Terms— Divisor Cayley Graph, total dominating set, total domination number.

1 INTRODUCTION

Berge[4] and Ore[12] are the first to introduce the concept of domination and they have contributed significantly to the theory of domination in graphs. E.J. Cockayne and S.T. Hedetniemi [6] published the first paper entitled "optimal domination in graphs". They were the first to use the notation $\gamma(G)$ for the domination number of a graph, which subsequently became the accepted notation. Allan and Laskar [1], Cockayne and Hedetniemi [6], Arumugam [3], Sampath Kumar [13] and others have contributed significantly to the theory of dominating sets and domination numbers. An introduction and an extensive overview on domination in graphs and related topics are given by Haynes et al. [8]. In the sequel edited by Haynes, Hedetniemi and Slater [9], several authors presented a survey of articles in the wide field of domination in graphs. Cockayne, C.J et.al [5] introduced the concept of total domination in graphs and studied extensively. The applications of both domination and total domination are widely used in Networks.

In this chapter we discuss total dominating sets of Divisor Cayley Graphs. Here we have obtained these sets for various values of n . In certain cases we could get total dominating sets. In other cases, we are unable to give the proofs to find the minimal sets by the technique we adopted to find these sets. This is because, the divisors depend on the number n and unless we give the value for n, we are not known the divisors and hence the elements in S^*.

Hence we have devised an algorithm which finds all minimal total dominating sets for all n. Our algorithm finds all closed and open neighborhood sets respectively and then finds a minimum number of these sets which cover all the vertices of G(Z_n, D). Each theorem is strengthened with examples and the Algorithm is also illustrated for different values of n.

2.1 Total dominating sets of Divisor Cayley Graphs

Let G be a graph without isolated vertices. Then a total dominating set T is a subset of V(G) such that every vertex in V is adjacent to some vertex in T.

A total dominating set with minimum cardinality is called a minimum total dominating set and the cardinality of a minimum total dominating set is called the total domination number of G and is denoted by γ_t (G).

In this section we find he total dominating sets of G (Z_n, D) for various values of n and find the values for this parameter.

Theorem 2.1..1: Suppose n is a prime of the form 4m +1.Then T= {2, 3, 6, 7, 10, 11, ..., n-2, n-1} is a total dominating set of G (Z_n, D).

Dept of Applied Mathematics, SPMVV, Tirupati, A.P., India
Email: maherahul.55@gmail.com

ISSN 2229-5518

Proof: Let n be a prime. Consider G (Z_n, D).Then the graph G (Z_n, D) is an outer Hamilton Cycle.

Let T = { 2, 3, 6, 7, 10, 11, …, n-2, n-1}.

Since G (Z_n, D) is an outer Hamilton cycle, every vertex in T is adjacent with its preceding and succeeding vertices. i.e., every vertex in V is adjacent to atleast one vertex of T. Therefore, T becomes a dominating set.

Since the vertices in T are pair wise viz., (2,3),(6,7),….,(n-2,n-1) are adjacent it follows that T is a total dominating set of G (Z_n, D).

Clearly we can see that no proper subset of T is a total dominating set of G (Z_n, D) so that T is a minimal total dominating set.

Hence |T| = k is the total domination number of G (Z_n, D).

Example 2.1.2: Consider G (Z_5, D) for n = 5 which is of the form 4m+1.

Then S ={1,5} and S^* = {1,4}.

The graph of G (Z_5, D) is given below.

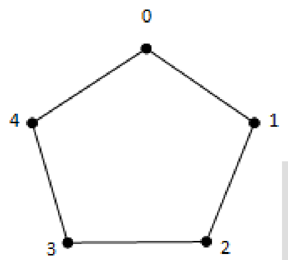

Fig . 1 : G (Z_5, D)

Further the graph G (Z_5, D) is 2 – regular. Also G (Z_5, D) is an Outer Hamilton cycle.

Let T = {2,3,4}. We observe that the set T = {2,3,4} dominates all the vertices in V so that T becomes a total dominating set of G(Z_n, D). Further, this T is a minimal because if we delete one vertex from this set then we can easily see that the remaining vertices cannot dominate the vertices of G(Z_5, D).

Therefore, the total domination number of G (Z_5, D) is |T| = 4.

Theorem 2.1.3: Suppose n is a prime of the form 4m +3. Then T = { 2, 3, 6, 7, 10, 11, ….., n-1, n } is a total dominating set of G (Z_n, D).

Proof: Let n be a prime. Consider G (Z_n, D).Then the graph G (Z_n, D) is an outer Hamilton Cycle.

Let T = { 2, 3, 6, 7, 10, 11, …, n-1, n}.

Since G (Z_n, D) is an outer Hamilton cycle, every vertex in T is adjacent with its preceding and succeeding vertices. i.e., every vertex in V is adjacent to atleast one vertex of T.

Therefore, T becomes a dominating set.

Since the vertices in T are pair wise viz., (2,3),(6,7),….,(n-2,n-1) are adjacent it follows that T is a total dominating set of G (Z_n, D). Clearly we can see that no proper subset of T is a total dominating set of G (Z_n, D) so that T is a minimal total dominating set.

Hence |T| = k is the total domination number of G (Z_n, D).

Example 2.1.4: Consider G (Z_7, D) for n = 7 which is of the form 4m + 3.

Then S ={1,7} and S^* = {1,6}.

The graph of G (Z_7, D) is given below.

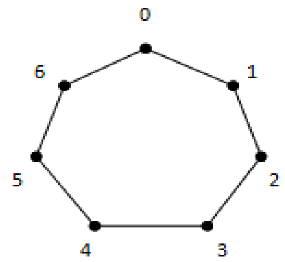

Fig. 2 : G (Z_7, D)

Further the graph G (Z_7, D) is 2 – regular. Also G (Z_7, D) is an Outer Hamilton Cycle.

Let T = {2,3,5,6}. We observe that the set T = {2,3,5,6} dominates all the vertices in V so that T becomes a total dominating set of G(Z_n, D). Further, this T is minimal because if we delete one vertex from this set then we can easily see that the remaining set cannot be a total dominating set of G(Z_7, D) .

Therefore, the total domination number of G (Z_7, D) is |T| = 3.

Remark : The total domination number of G(Z_n, D) is 1 if n = 2,3,4 and 6. The symmetric subset S^* contains all the vertices of G(Z_n, D) for all value of n and hence the difference of any two vertices is in S^*. Thus every vertex is adjacent to all the vertices of G(Z_n, D) so that it is complete. So, every single time vertex is total dominating set and hence the total domination number is 1.

Theorem 2.1.5 : If n is not a prime and n= 2p where p > 5 is a prime or n= 5^m where m > 1 then T= {2, 3, 6, 7, 10, 11, …, n-2, n-1} is a total dominating set of G (Z_n, D).

Proof: Suppose n be not a prime. Consider the graph G (Z_n, D).

Case 1: Let n = 2p where p > 5 is a prime.

In this case S = {1, 2, p, 2p} and S^* = {1, 2, p, 2p – 1,2p – p, 2p - 2 }.

The graph is $|S^*|$ - regular. i.e., the graph is 6 – regular.

Let T = {2, 3, 6, 7, 10, 11, ….., n-2, n-1}. We now show that T is a total dominating set of G (Z_n, D).

The set of vertices adjacent with the vertices of T are given below.

0→ 1, 2, p, 2p – 1,2p – p, 2p – 2

2→ 3, 4, p + 2, 2p + 1,2p – p + 2, 2p .

3→ 4, 5, p + 3, 2p + 2,2p – p + 3, 2p + 1.

⋮

n-2→ n – 1, n, p + n – 2, 2p + n – 3, 2p + n – 2 – p, 2p – n – 4.

n-1→ n, n + 1, p + n – 1, 2p + n – 4, 2p + n – 3 – p, 2p – n – 3.

Here we observe that the vertices {2, 3, 6, 7, 10, 11, …, n-2, n-1} are dominating the vertices 3, 4, 7, 8, …., n-1, 0 , respectively.

If we give the value for p then the rest of the vertices which are dominated by {2, 3, 6, 7, 10, 11, …., n-2, n-1} will also be found and these vertices are all the vertices of G(Z_n, D).

Likewise, the vertices in T dominate the vertices of the graph G(Z_n, D) and they are also adjacent among themselves pair wise as given in Theorem 4.2.1.

Thus T becomes a total dominating set.

But here T is not minimal.

Case 2: Let n = p^m where m > 1 and p = 5.

Consider the graph G (Z_n, D).

In this case, S = {{1, p, p^2, … p^m} and

S^* = { 1, p, p^2, …., p^{m-1}, p^m –p, p^m-p^2, p^m-p^3, …., p^m – p^{m-1}}

The graph is |S*| - regular.

Let T = {2, 3, 6, 7, 10, 11, …., n-2, n-1}. We now show that T is a total dominating set of G (Z_n, D).

The set of vertices adjacent with the vertices of T are given below.

0→ 1, p, p^2,…., p^{m-1}, p^m- p, p^m- p^2, p^m- p^3, .., p^m- p^{m-1}.

2→ 3, p+2, p^2+ 2, …, p^{m-1}+2, p^m- p+2, p^m- p^2+2, p^m- p^3+2, …, p^m- p^{m-1}+2.

3→ 4, p+3, p^2+3, …, p^{m-1}+3, p^m- p+3, p^m- p^2+3, p^m- p^3+3, …, p^m- p^{m-1}+3.

⋮

n-2 → n-1, p+ n-2, p^2+n-2, …, p^{m-1}+n-2, p^m- p+n-2, p^m- p^2+n-2, p^m- p^3+n-2, …, p^m- p^{m-1}+n-2

n-1→ 0, p+ n-1, p^2+n-1,…, p^{m-1}+n-1, p^m- p+n-1, p^m- p^2+n-1, p^m- p^3+n-1, …, p^m- p^{m-1}+n-1

Here we observe that the vertices {2, 3, 6, 7, 10, 11, …, n-2, n-1} are dominating the vertices 3, 4, 7, 8, …., n-1, 0 respectively.

If we replace p = 5 and give value for m, then the set of vertices which are dominated by {2, 3, 6, 7, 10, 11, …., n-2, n-1} will be found and this set is nothing but the vertex set of G (Z_n, D).

Likewise, the vertices in T dominate the vertices of the graph G (Z_n, D) and hence T is a total dominating set of G (Z_n, D).

But T is not minimal.

Example 2.1.6 : Let n=14.

Then S = {1, 2, 7, 14} and S^* = {1, 2, 7, 12, 13}.

The graph of G (Z_{14}, D) is given below.

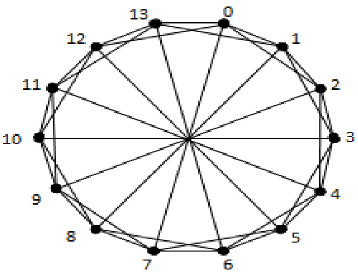

Fig .3 : G (Z_{14}, D)

Now T = {2, 3, 6, 7, 10, 11, 12, 13}.

The vertices adjacent with the vertices of T are given by

2 → 3, 4, 9, 0, 1

3 → 4, 5, 10, 1, 2

6 → 7, 8, 13, 4, 5

7 → 8, 9, 0, 5, 6

10 → 11, 12, 3, 8, 9

11 → 12, 13, 4, 9, 10

12 → 13, 0, 5, 10, 11

13 → 0, 1, 6, 11, 12.

The set T = {2, 3, 6, 7, 10, 11, 12, 13} dominates the vertices 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. Also the vertices in T are adjacent pair wise.

Hence T = {2, 3, 6, 7, 10, 11, 12, 13} becomes a total dominating set of G (Z_{14}, D).

Since the graph is 5 – regular, we should have |T| ≥ 3. But here we have obtained |T| = 8. That is T is not minimal.

To get minimal total dominating sets, we give the following algorithm which finds the minimal total dominating sets of G (Z_n, D), where n = 2p for p > 5 is a prime.

If we run the algorithm, then the minimal total dominating set obtained for this graph is T = { 0, 7, 2, 9} whose cardinality is 4.

This algorithm is run by finding a minimum set of open neighbourhood sets which cover all the vertices of G (Z_n, D).

Example 2.1.7 : Let n = 25.

Then S = {1, 5, 25}, S^* = {1, 5, 20, 24}

ISSN 2229-5518

The graph of G (Z_{25}, D) is given below.

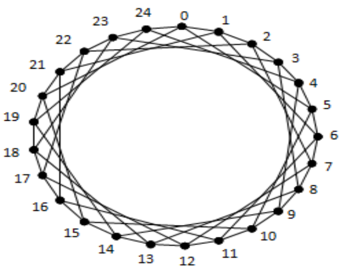

Fig .4 : G (Z_{25}, D)

Let T= { 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 24}.
The vertices adjacent with the vertices of T are

$2 \rightarrow 3, 7, 22, 1$
$3 \rightarrow 4, 8, 23, 2$
$6 \rightarrow 7, 11, 1, 5$
$7 \rightarrow 8, 12, 2, 6$
$10 \rightarrow 11, 15, 5, 9$
$11 \rightarrow 12, 16, 6, 0$
$14 \rightarrow 15, 19, 9, 3$
$15 \rightarrow 16, 20, 10, 14$
$18 \rightarrow 19, 23, 13, 17, 21$
$19 \rightarrow 20, 24, 14, 18$
$22 \rightarrow 23, 2, 17, 21$
$23 \rightarrow 24, 3, 18, 22$
$24 \rightarrow 0, 4, 19, 23$

Here T = { 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 24} dominate all the vertices of G (Z_{25}, D). Hence this set becomes a total dominating set of G (Z_{25}, D). Since G (Z_{25}, D) is 4 – regular, we should have $|T| \geq 7$.Here we have obtained $|T| = 13$.

Hence T is not minimal.

To get minimal total dominating sets, we run the Algorithm and the minimal total dominating set obtained for G (Z_{25}, D) is { 0, 1, 8, 9, 17, 16, 18} whose cardinality is 7.

The following Algorithm finds minimal total dominating sets of G(Z_n, D) for all values of n except when n is a prime. Because when n is a prime, the graph becomes an outer Hamilton cycle and hence the total dominating sets are found easily.

Algorithm – TDS - DCG:

INPUT : Enter a number
OUTPUT: Minimal total dominating sets
STEP 1 : Enter a number n
STEP 2 : IF n IS NOT EQUAL TO NULL
GOTO STEP 3
ELSE GOTO STEP 12

STEP 3 : METHOD setofdivisiors($number){
INITIALIZE ARRAY $divisorset
FOR EACH $i,$i=1,......$i<=$number
DO THE FOLLOWING
INITIALIZE $modvalASSIGN $number % $i TO $modval
IF $modval EQUAL TO 0ASSIGN $i TO $divisorset[]
END IF
RETURN $divisorset
END FOR

STEP 4 : Find set of divisors
CALL METHOD setofdivisiors(param)
INITIALIZE VARIABLE $result ASSIGN
COUNT OF(setofdivisiors(param))
INITIALIZE VARIABLE $setOfDivisorsArray ASSIGN
setofdivisiors(param)
PRINT "Set of divisors ={";
FOR EACH $i,$i=0,......$i<param
INITIALIZE ARRAY $setOfDivisorsArrayTemp[]
ASSIGN
$setOfDivisorsArray[$i];
PRINT $setOfDivisorsArray[$i];
IF $i NOT EQUAL TO $result-1
PRINT ",";
END IF
END FOR
PRINT "}"

STEP 5 : Find Symmetric sub set
INITIALIZE VARIABLE $setOfDivisorsCount ASSIGN
COUNT
OF($setOfDivisorsArrayTemp);
INITIALIZE ARRAY $firstTempArray
INITIALIZE ARRAY $secondTempArray
FOR EACH $j,$j=0,..........$j<$setOfDivisorsCount
IF $setOfDivisorsArrayTemp[$j] NOT EQUAL TO
$number
IF $setOfDivisorsArrayTemp[$j] EQUAL TO $number
ELSE
ASSIGN VALUE $setOfDivisorsArrayTemp[$j] TO
$firstTempArray[]
ASSIGN VALUE ($number-
$setOfDivisorsArrayTemp[$j]) TO
$secondTempArray[]
END IF
END IF
END FOR
MERGE $firstTempArray AND $secondTempArray
array_merge($firstTempArray, $secondTempArray)
INITIALIZE VARIABLE $mergeFSTemps

```
REMOVE DUPLICATE VALUES USING
array_unique(array_merge($firstTempArray,
$secondTempArray))
ASSIGN TO $mergeFSTemps
SORT ARRAY asort($mergeFSTemps);
INITIALIZE VARIABLE $keynum
ASSIGN 0 TO $keynum
WHILE (list($key, $val) = EACH($mergeFSTemps)) {
INITIALIZE ARRAY $symmetricSubSet
ASSIGN VALUE $val TO $symmetricSubSet[]
INCREMENT $keynum++;
END WHILE
INITIALIZE VARIABLE $symmetricSubSetCount
ASSIGN COUNT OF count($symmetricSubSet)
        TO $symmetricSubSetCount
PRINT "S* ={"
FOR EACH $m,$m=0,..........$m<$symmetricSubSetCount
PRINT $symmetricSubSet[$m];
IF $m NOT EQUAL TO $symmetricSubSetCount-1
PRINT ",";
END IF
END FOR
PRINT  "}"
```

STEP 6 : Find Neighbourhood sets of n

```
FOR EACH $nsa,$nsa=0,..........$nsa<$number
FOR EACH $nsb,$nsb=0,..........$nsb<$symmetricSubSet
INITIALIZE MULTI DIMENSIONAL ARRAY
        $neighbourhoodSetArray
ASSIGN $symmetricSubSet[$nsb] TO
 $neighbourhoodSetArray[$nsa][$nsb]
ASSIGN $symmetricSubSet[$nsb]+1 TO
$symmetricSubSet[$nsb]
IF $symmetricSubSet[$nsb] EQUAL TO  $number
ASSIGN 0 TO $symmetricSubSet[$nsb]
END IF
END FOR
END FOR
PRINT "Neighbourhood sets of n are"
FOR EACH $ns,$ns=0,...........$ns<$number
PRINT  "N[$ns]={"
FOREACH
 $nbs,$nbs=0,...........$nbs<count($neighbourhoodSetArray[$ns]
PRINT $neighbourhoodSetArray[$ns][$nbs];
IF $nbs EQUAL TO
count($neighbourhoodSetArray[$ns])-1
PRINT ""
ELSE
PRINT ","
END FOR
PRINT "}";
END FOR
```

STEP 7 : PRINT "Consider N[0] ={";

```
FOR EACH
$m,$m=0,..........$m<count($neighbourhoodSetArray[0])
INITIALIZE ARRAY $nof0array
ASSIGN $neighbourhoodSetArray[0][$m] TO
$nof0array[]
PRINT $neighbourhoodSetArray[0][$m];
IF $m NOT EQUAL TO
count($neighbourhoodSetArray[0])-1
PRINT ",";
END IF
END FOR
PRINT "}"
```

STEP 8 : Find Uncovered vertices in N[0]

```
FOR EACH $m,$m=0,..........$m<$number
INITIALIZE ARRAY $notin
IF !in_array($m, $nof0array)
ASSIGN$m TO $notin[]
END IF
END FOR
PRINT "Uncovered vertices in N[0] are {";
FOR EACH $m,$m=0,..........$m<count($notin)
PRINT $notin[$m];
IF $m NOT EQUAL TO count($notin)-1)
PRINT ",";
END IF
END FOR
PRINT "}";
```

STEP 9 : Find Minimal total dominating set

```
INITIALIZE $nofzerocount
ASSIGN count($neighbourhoodSetArray[0]) TO
$nofzerocount
INITIALIZE $after_merge
ASSIGN $neighbourhoodSetArray[0] TO $after_merge
INITIALIZE VARIABLE $nsstart=1;
INITIALIZE ARRAY $selected_neigh_sets
REFERENCE OF GO TO a:
FOR EACH $nss,$nss=$nsstart,..........$nss<count
        ($neighbourhoodSetArray)
IF NOT IN ARRAY($nss, $selected_neigh_sets) AND
        count($notin)NOT EQUALTO NULL
INITIALIZE VARIABLE $result
ASSIGN array_intersect($neighbourhoodSetArray[$nss],
$notin) TO   $result
IF count($result) EQUAL TO  $nofzerocount
```

ISSN 2229-5518

```
MERGE $after_merge,$neighbourhoodSetArray[$nss] AND
ASSIGN TO $after_merge
ASSIGN array_diff($notin,$after_merge) TO $notin
ASSIGN $nss TO $selected_neigh_sets[]
ELSE
IF count($neighbourhoodSetArray)-1) EQUAL TO  $nss
ASSIGN $nsstart=1;
ASSIGN $nofzerocount=$nofzerocount-1;
GOTO a;
END IF
END IF
END IF
END FOR
PRINT "Minimal  total dominating set is  ={";
FOR EACH $m,$m=0,..........$m<count($selected_neigh_sets)
PRINT $selected_neigh_sets[$m];
IF $m NOT EQUAL TO count($selected_neigh_sets)-1)
PRINT ",";
END IF
END FOR
PRINT "}";
```

STEP 10: FIND the number of minimal total dominating sets

```
INITIALIZE $finalone
ASSIGN $selected_neigh_sets TO $finalone
PRINT "The number of minimal dominating sets of n are"
FOR EACH $nsone,$nsone=0,..........$nsone<$number
PRINT "{";
FOR EACH $nbsone,$nbsone=0,..........$nbsone<count($finalone)
IF $finalone[$nbsone] EQUAL TO $number$finalone[$nbsone]=0;
END IF
PRINT $finalone[$nbsone]
IF $nbsone EQUAL TO  (count($finalone)-1)
PRINT "";
ELSE
PRINT ",";
END IF
INCREMENT $finalone[$nbsone]++
END FOR
PRINT "}";
INITIALIZE ARRAY $tempfinalone
ASSIGN $finalone TO $tempfinalone[]
SORT $tempfinalone[$nsone]
SORT $selected_neigh_sets
IF count(array_diff($tempfinalone[$nsone], $selected_neigh_sets) EQUAL TO 0
BREAK;
ELSE
PRINT ",";
continue;
END IF
END FOR
```

STEP 11: PRINT "The total domination number is ".count($selected_neigh_sets)

STEP 12: PRINT "please enter a proper number"

STOP.

Theorem 2.1.8 : Let n be not a prime and $n \neq 2p$, for $p > 5$ is a prime and $n \neq 5^m$ for $m > 1$. If n is divisible by 5 then $T = \{rd_o \,/\, 0 \leq r \leq k\text{-}1$ where k is the largest positive integer such that $rd_o < n$ and $d_0 = 5\}$ is a total dominating set of $G(Z_n, D)$.

Proof: Let n be not a prime, $n \neq 2p$ for $p > 5$ is a prime and $n \neq 5^m$ for $m > 1$.

Then $n = 5m$ where $m \neq 5^r$ for $r > 1$.

So $n = 5\,(p_1^{\alpha_1} \cdot p_2^{\alpha_2}, \ldots, p_m^{\alpha_m})$ where $p_1, p_2, \ldots, p_m$ are primes and $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 1$.

In this case $S = \{1, 5, p_1, \ldots, p_1^{\alpha 1}, p_2, \ldots p_2^{\alpha 2}, .., p_m, .., p_m^{\alpha m}, p_1.\, p_2, p_1.\, p_2^2, .., p_1.\, p_2^{\alpha 2}, .., p_{m-1}.\, p_m^{\alpha m}, ,\ldots\ldots, p_{m-1}^{\alpha m-1} . p_m^{\alpha m}, \ldots, p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_m^{\alpha m}, \ldots, 5p_1, 5p_2, \ldots, 5p_m, \ldots, 5p_1.p_2, \ldots, 5p_{m-1}.p_m\}$.

$S^* = \{1, 5, p_1, \ldots, p_1^{\alpha 1}, p_2, \ldots p_2^{\alpha 2}, .., p_m, .., p_m^{\alpha m}, p_1.\, p_2, p_1.\, p_2^2, .., p_1.\, p_2^{\alpha 2}, .., p_{m-1}.\, p_m^{\alpha m}, \ldots\ldots, p_{m-1}^{\alpha m-1} . p_m^{\alpha m}, .., \ldots, p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_m^{\alpha m}, 5p_1, 5p_2, \ldots, 5p_m, \ldots, 5p_1.p_2, \ldots, 5p_{m-1}.p_m, n-1, n-p_1, \ldots\ldots, n - p_{m-1}^{\alpha m-1} . p_m^{\alpha m}, n - 5p_1, \ldots, n - 5p_{m-1}.p_m\}$.

Le $T = \{rd_o \,/\, 0 \leq r \leq k\text{-}1$ where k is the largest positive integer such that $rd_o < n$ and $d_0 = 5\}$

The following set of vertices are adjacent with the vertices in T.

$0 \rightarrow 1, 5, p_1, \ldots, p_1^{\alpha 1}, p_2, \ldots p_2^{\alpha 2}, .., p_m, .., p_m^{\alpha m}, p_1.\, p_2, p_1.\, p_2^2, .., p_1.\, p_2^{\alpha 2}, .., p_{m-1}.\, p_m^{\alpha m}, ,\ldots\ldots, p_{m-1}^{\alpha m-1} . p_m^{\alpha m}, ..,\ldots, p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_m^{\alpha m}, , 5p_1, 5p_2, \ldots, 5p_m,$

ISSN 2229-5518

…., $5p_1.p_2, \ldots, 5p_{m-1}.p_m$, $n - 1$, $n - p_1$, ……., $n - p_{m-1}^{\alpha m-1}$. $p_m^{\alpha m}$, $n - 5p_1$, ………, $n - 5p_{m-1}.p_m$

$5 \rightarrow 6, 11, p_1 + 5, \ldots, p_1^{\alpha 1} + 5, p_2 + 5, \ldots p_2^{\alpha 2} + 5, \ldots, n - 5p_{m-1}.p_m - n + 5$.

$\vdots$

Here we observe that the vertices {0,5,10,.. }are dominating the vertices 1,6,11,…

If we give values for $p_1, p_2, \ldots, p_m$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ then the set of vertices which are dominated by {0, 5, 10, 15, …} will be found and this set is nothing but the vertex set of G (Z_n, D).

Likewise, the vertices in T dominate all the vertices of the graph G (Z_n, D) and they are also adjacent among themselves. Hence T is a total dominating set of G (Z_n, D).

But T is not minimal.

Example 2.1.9 : Let n = 30.

Then S ={1,2,3,5,6,10,15,30} and S* ={1,2,3,5,6,10,15,20,24,25,27,28,29}.

The graph of G (Z_{30}, D) is given below.

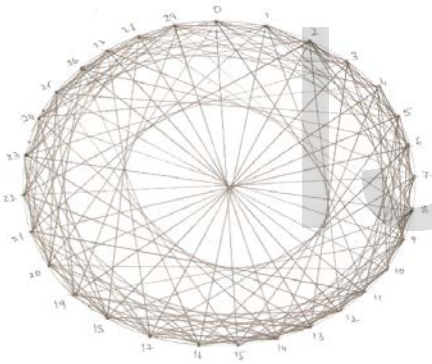

Fig. 5 : G (Z_{30}, D)

Further the graph G (Z_{30}, D) is 13– regular.

Let T = {0,5,10,15,20,25}.

The vertices adjacent with the vertices of T are

$0 \rightarrow$ 1,2,3,5,6,10,15,20,24,25,27,28,29

$5 \rightarrow$ 6,7,8,10,11,15,20,25,29,0,2,3,4

$10 \rightarrow$ 11,12,13,15,16,20,25,0,4,5,7,8,9

$15 \rightarrow$ 16,17,18,20,21,25,0,5,9,10,12,13,14

$20 \rightarrow$ 21,22,23,25,26,0,5,10,14,15,17,18,19

$25 \rightarrow$ 26,27,28,0,1,5,10,15,19,20,22,23,24

Here T = {0,5,10,15,20,25} dominates all the vertices of G (Z_{30}, D) and the vertices in T are also adjacent among themselves. Hence this set becomes a total dominating set of G (Z_{30}, D). Since G (Z_{30}, D) is 13 – regular, we should have $|T| \geq 3$. Here we have obtained $|T| = 6$. Hence T is not minimal.

To get minimal total dominating sets, we run the Algorithm and the minimal total dominating set obtained for G (Z_{30}, D) is {0,6,16,3}whose cardinality is 4.

Theorem 2.1.10 : For other values of n where n is not a prime, $n \neq 2p$, $p > 5$, $n \neq 5^m$, $m > 1$, n is not divisible by 5, the minimal total dominating sets of G(Z_n, D) are obtained by the Algorithm – TDS – UDCG.

For n = 12, n = 21 the minimal total dominating sets are given in Algorithm – illustrations.

The given Algorithm is developed by using the PHP software (server scripting language) and the following illustrations are obtained by simply giving the value for n.

2.2 Algorithm - Illustrations

1. Let n = 2p where p > 5 is a prime.

Suppose n = 14.

S = {1,2,7,14}

S* = {1,2,7,12,13}

Neighbourhood sets of G(Z_{14} , D) are

N(0) ={1,2,7,12,13}

N(1) ={2,3,8,13,0}

N(2) ={3,4,9,0,1}

N(3) ={4,5,10,1,2}

N(4) ={5,6,11,2,3}

N(5) ={6,7,12,3,4}

N(6) ={7,8,13,4,5}

N(7) ={8,9,0,5,6}

N(8) ={9,10,1,6,7}

N(9) ={10,11,2,7,8}

N(10) ={11,12,3,8,9}

N(11) ={12,13,4,9,10}

N(12) ={13,0,5,10,11}

N(13) ={0,1,6,11,12}

Minimal total dominating set is ={0,7,2,9}

The number of minimal total dominating sets of G(Z_{14} , D) are {0,7,2,9},{1,8,3,10},{2,9,4,11},{3,10,5,12},{4,11,6,13},{5,12,7,0}, {6,13,8,1},{7,0,9,2},{8,1,10,3},{9,2,11,4},{10,3,12,5},{11,4,13,6}, {12,5,0,7},{13,6,1,8}.

The total domination number is 4.

2. Let $n = p^m$ where m > 1 and p = 5.

Suppose n = 25.

S = {1,5,25}

S* ={1,5,20,24}

Neighbourhood sets of G(Z_{25} , D) are

N(0) ={1,5,20,24}

N(1) ={2,6,21,0}
N(2) ={3,7,22,1}
N(3) ={4,8,23,2}
N(4) ={5,9,24,3}
N(5) ={6,10,0,4}
N(6) ={7,11,1,5}
N(7) ={8,12,2,6}
N(8) ={9,13,3,7}
N(9) ={10,14,4,8}
N(10) ={11,15,5,9}
N(11) ={12,16,6,10}
N(12) ={13,17,7,11}
N(13) ={14,18,8,12}
N(14) ={15,19,9,13}
N(15) ={16,20,10,14}
N(16) ={17,21,11,15}
N(17) ={18,22,12,16}
N(18) ={19,23,13,17}
N(19) ={20,24,14,18}
N(20) ={21,0,15,19}
N(21) ={22,1,16,20}
N(22) ={23,2,17,21}
N(23) ={24,3,18,22}
N(24) ={0,4,19,23}

Minimal total dominating set is ={0,1,8,9,17,16,18}

The number of minimal total dominating sets of G(Z_{25} , D) are {0,1,8,9,17,16,18},{1,2,9,10,18,17,19},{2,3,10,11,19,18,20}, {3,4,11,12,20,19,21}, {4,5,12,13,21,20,22},{5,6,13,14,22,21,23}, {6,7,14,15,23,22,24}, {7,8,15,16,24,23,0}, {8,9,16,17,0,24,1}, {9,10,17,18,1,0,2}, {10,11,18,19,2,1,3},{11,12,19,20,3,2,4}, {12,13,20,21,4,3,5},{13,14,21,22,5,4,6},{14,15,22,23,6,5,7}, {15,16,23,24,7,6,8}, {16,17,24,0,8,7,9}, {17,18,0,1,9,8,10}, {18,19,1,2,10,9,11},{19,20,2,3,11,10,12}, {20,21,3,4,12,11,13}, {21,22,4,5,13,12,14},{22,23,5,6,14,13,15},{23,24,6,7,15,14,16}, {24,0,7,8,16,15,17}.

The total domination number is 7.

3. Let n = 5m where m ≠ 5^r for r > 1.

Suppose n = 30.

S = {1,2,3,5,6,10,15,30}

S* ={1,2,3,5,6,10,15,20,24,25,27,28,29}

Neighbourhood sets of G(Z_{30} , D) are

N(0) ={1,2,3,5,6,10,15,20,24,25,27,28,29}
N(1) ={2,3,4,6,7,11,16,21,25,26,28,29,0}
N(2) ={3,4,5,7,8,12,17,22,26,27,29,0,1}
N(3) ={4,5,6,8,9,13,18,23,27,28,0,1,2}
N(4) ={5,6,7,9,10,14,19,24,28,29,1,2,3}
N(5) ={6,7,8,10,11,15,20,25,29,0,2,3,4}
N(6) ={7,8,9,11,12,16,21,26,0,1,3,4,5}
N(7) ={8,9,10,12,13,17,22,27,1,2,4,5,6}
N(8) ={9,10,11,13,14,18,23,28,2,3,5,6,7}
N(9) ={10,11,12,14,15,19,24,29,3,4,6,7,8}
N(10) ={11,12,13,15,16,20,25,0,4,5,7,8,9}
N(11) ={12,13,14,16,17,21,26,1,5,6,8,9,10}
N(12) ={13,14,15,17,18,22,27,2,6,7,9,10,11}
N(13) ={14,15,16,18,19,23,28,3,7,8,10,11,12}
N(14) ={15,16,17,19,20,24,29,4,8,9,11,12,13}
N(15) ={16,17,18,20,21,25,0,5,9,10,12,13,14}
N(16) ={17,18,19,21,22,26,1,6,10,11,13,14,15}
N(17) ={18,19,20,22,23,27,2,7,11,12,14,15,16}
N(18) ={19,20,21,23,24,28,3,8,12,13,15,16,17}
N(19) ={20,21,22,24,25,29,4,9,13,14,16,17,18}
N(20) ={21,22,23,25,26,0,5,10,14,15,17,18,19}
N(21) ={22,23,24,26,27,1,6,11,15,16,18,19,20}
N(22) ={23,24,25,27,28,2,7,12,16,17,19,20,21}
N(23) ={24,25,26,28,29,3,8,13,17,18,20,21,22}
N(24) ={25,26,27,29,0,4,9,14,18,19,21,22,23}
N(25) ={26,27,28,0,1,5,10,15,19,20,22,23,24}
N(26) ={27,28,29,1,2,6,11,16,20,21,23,24,25}
N(27) ={28,29,0,2,3,7,12,17,21,22,24,25,26}
N(28) ={29,0,1,3,4,8,13,18,22,23,25,26,27}
N(29) ={0,1,2,4,5,9,14,19,23,24,26,27,28}

Minimal total dominating set is ={0,6,16,3}

The number of minimal total dominating sets of G(Z_{30} , D) are {0,6,16,3},{1,7,17,4},{2,8,18,5},{3,9,19,6},{4,10,20,7},{5,11,21,8}, {6,12,22,9}, {7,13,23,10},{8,14,24,11},{9,15,25,12},{10,16,26,13}, {11,17,27,14},{12,18,28,15}, {13,19,29,16},{14,20,0,17},{15,21,1,18}, {16,22,2,19},{17,23,3,20},{18,24,4,21}, {19,25,5,22},{20,26,6,23}, {21,27,7,24}, {22,28,8,25},{23,29,9,26}, {24,0,10,27},{25,1,11,28}, {26,2,12,29},{27,3,13,0},{28,4,14,1},{29,5,15,2}.

The total domination number is 4.

4. Let n =12

S ={1,2,3,4,6,12}

S* ={1,2,3,4,6,8,9,10,11}

Neighbourhood sets of G(Z_{12} , D) are

N(0) ={1,2,3,4,6,8,9,10,11}
N(1) ={2,3,4,5,7,9,10,11,0}
N(2) ={3,4,5,6,8,10,11,0,1}
N(3) ={4,5,6,7,9,11,0,1,2}
N(4) ={5,6,7,8,10,0,1,2,3}
N(5) ={6,7,8,9,11,1,2,3,4}

N(6) ={7,8,9,10,0,2,3,4,5}
N(7) ={8,9,10,11,1,3,4,5,6}
N(8) ={9,10,11,0,2,4,5,6,7}
N(9) ={10,11,0,1,3,5,6,7,8}
N(10) ={11,0,1,2,4,6,7,8,9}
N(11) ={0,1,2,3,5,7,8,9,10}
Minimal total dominating set is {0,1}
The minimal total dominating sets of G(Z_{12} , D) are {0,1},{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{8,9},{9,10},{10,11},{11,0}.
The total domination number is 2.

5. Let n =21
S ={1,3,7,21}
S* ={1,3,7,14,18,20}
Neighbourhood sets of G(Z_{21} , D) are
N(0) ={1,3,7,14,18,20}
N(1) ={2,4,8,15,19,0}
N(2) ={3,5,9,16,20,1}
N(3) ={4,6,10,17,0,2}
N(4) ={5,7,11,18,1,3}
N(5) ={6,8,12,19,2,4}
N(6) ={7,9,13,20,3,5}
N(7) ={8,10,14,0,4,6}
N(8) ={9,11,15,1,5,7}
N(9) ={10,12,16,2,6,8}
N(10) ={11,13,17,3,7,9}
N(11) ={12,14,18,4,8,10}
N(12) ={13,15,19,5,9,11}
N(13) ={14,16,20,6,10,12}
N(14) ={15,17,0,7,11,13}
N(15) ={16,18,1,8,12,14}
N(16) ={17,19,2,9,13,15}
N(17) ={18,20,3,10,14,16}
N(18) ={19,0,4,11,15,17}
N(19) ={20,1,5,12,16,18}
N(20) ={0,2,6,13,17,19}
Minimal total dominating set is ={0,1,9,10,2}
The minimal total dominating sets of G(Z_{21} , D) are {0,1,9,10,2},{1,2,10,11,3},{2,3,11,12,4},{3,4,12,13,5},{4,5,13,14,6},
{5,6,14,15,7},{6,7,15,16,8},{7,8,16,17,9},{8,9,17,18,10},{9,10,18,19,11},
{10,11,19,20,12},{11,12,20,0,13},{12,13,0,1,14},{13,14,1,2,15},
{14,15,2,3,16},{15,16,3,4,17},{16,17,4,5,18},{17,18,5,6,19},{18,19,6,7,20}, {19,20,7,8,0},{20,0,8,9,1},
The total domination number is 5.

2. References

[1] Allan, R.B. and Laskar, R.C.On domination and independent domination numbers of a graph, Discrete Math, 23 (1978), 73-76.

[2] Apostol, M.Tom.Introduction to Analytic Number theory, Springer International Student Edition, (1989).

[3] Arumugam, S. Uniform Domination in graphs, National Seminar on graph theory and its applications, January (1983).

[4] Berge, C.The Theory of Graphs and its Applications, Methuen, London (1962).

[5] Cockayne, C.J., Dawes, R.M. and Hedetniemi, S.T.Total domination in graphs, Networks, 10 (1980), 211-219.

[6] Cockayne, C.J., and Hedetniemi, S.T.Towards a theory of domination in graphs, Networks, 7 (1977), 247 – 261.

[7] Harary, F.Graph Theory, Addison –Wesley, Reading, MA, (1969).

[8] Haynes, T.W., Hedetniemi, S.T. and Slater, P.J.Fundamentals of domination in graphs, Marcel Dekker, Inc., New York (1998).

[9] Haynes, T.W., Hedetniemi, S.T. and Slater, P.J.Domination in graphs: Advanced Topics, Marcel Dekker, Inc., New York (1998).

[10] Madhavi, L.Studies on domination parameters and enumeration of cycles in some Arithmetic Graphs, Ph. D. Thesis, submitted to S.V. University, Tirupati, India, (2002).

[11] Nathanson, B. Melvyn Connected components of arithmetic graphs,Monat. fur. Math, 29 (1980), 219 – 220.

[12] Ore, O.Theory of Graphs, Amer. Math. Soc. Colloq. Publ. vol. 38.Amer. Math. Soc., Providence, RI, (1962).

[13] Sampath Kumar, E.On some new domination parameters of a graph. A survey. Proceedings of a Symposium on Graph Theory and Combinatorics, Kochi, Kerala, India, 17 – 19 May (1991), 7 – 13.

[14] Sujatha, K.Sudies on Domination Parameters and Cycle Structure of Cayley Graphs associated with some Arithmetical Functions, Ph. D. thesis, S.V.University, 2008.

[15] Vizing, V.G.Vertex colorings with given colors, Metody Diskret. Analiz. (in Russian) 29:3-10,1976.